Department of Mathematics
Comprehensive Examination - Option I
2018 Spring

Algebra

1. Let G and G be groups and $\varphi : G \to G$ be a surjective group homomorphism.

   (a) Prove that the kernel K of φ is a subgroup of G and normal in G.

   (b) Define $\Phi : G/K \to G$ by $\Phi(gK) = \varphi(g) \quad gK \quad G/K$.
   Prove that Φ is a well-defined group isomorphism from G/K onto G .

2. Let R be a commutative ring with multiplicative identity 1. Prove:

   (a) $a \quad R$, $a = \{ar : r \quad R\}$ is an ideal of R containing a.

   (b) If the only ideal of R is $\{0\}$ and R, then R is a field.

3. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coe□cients in variable x.

   (a) Prove that 3 and x are irreducible polynomials in $\mathbb{Z}[x]$.

   (b) Prove or disprove that the ideal $I = \ 3, x \ = \{3f(x) + xg(x) : f(x), g(x) \quad \mathbb{Z}[x]\}$ is a principal ideal.

4. Let V and W be vector spaces of field F, $T : V \to W$ be an injective linear transformation, and $\{v_1, v_2, \cdots, v_k\}$ be a subset of V for some postive integer k. Prove: $\{v_1, v_2, \cdots, v_k\}$ is linearly independent in V if and only if $\{T(v_1), T(v_2), \cdots, T($is a ais u(and)08-14]

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Real Analysis

1. Given that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$

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Numerical Analysis

1. Consider the equation $x^2 - 3\ln(x) - 4 = 0$. (Note: For this problem, you may not use any graphing or root finding capabilities of your calculator.)

   (a) Prove that the equation has exactly two solutions.

4. The latest *Game of Thrones* book has just been published, and Amazon wants to make sure that all its warehouses are well stocked so that orders can be filled in a timely manner. Suppose there are four factories in North Carolina, producing, respectively,

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Probability

1. The probability of a telemarketing sales representative making a sale on a customer call is 0.10. Find the probability that
   (a) No sales are made in 10 calls.
   (b) More than 2 sales are made in 20 calls.

   The sales representatives are required to achieve a mean of at least 5 sales each day.

   (c) Find the least number of calls each day a representative should make to achieve this requirement.
   (d) Calculate the least number of calls that need to be made by a representative for the probability of at least 1 sale to exceed 0.95.

2. Let the random variable X be distributed as *Uniform(0,1)*.

$$f_X(x) = 1 \quad 0 < x < 1$$
$$= 0 \quad \text{otherwise}$$

   Let $Y = -2\ln(X)$.

   (a) Determine the distribution function of Y.
   (b) Determine the density function of Y.
   (c) Determine the moment generation function of Y.
   (d) If $Y_1, Y_2, \cdots, Y_n$ are independently distributed as Y, and $Z = Y_1 + Y_2 + \cdots + Y_n$, determine the moment generating function (MGF) of the random variable Z.

3. Suppose that X is the number of heads in four flips of a coin. Let Y be the random variable $X - 2$, the difference between X and its expected value.
   (a) Compute $E(Y)$. Does it effectively measure how much we expect to see X deviate from its expected value?
   (b) Compute variance of X, $V(X)$.
   (c) What is the sum of the variances for four independent trials of one flip of a coin? (hint: first find the variance for the number of heads in one flip of a coin).
   (d) If we want to be 95% sure that the number of heads in n flips of a coin is within $\pm 1\%$ of the expected value, how big does n have to be?

4. When coded messages are sent there are sometimes errors in the transmission. In particular, Morse code uses “dots” and “dashes,” which are known to occur in the proportion of 3:4. This means that for any given symbol

$$P(\text{dot sent}) = 3/7 \qquad P(\text{dash sent}) = 4/7$$

Suppose there is some interference on the transmission line, and with probability $\frac{1}{8}$ a sent dot is mistakenly received as a dash, and a sent dash is mistakenly received as a dot with the same probability.

(a) Specify the probability P(dash received|dot sent) and P(dot received|dash sent).

(b) Compute the probability P(dot received).

(c) Compute the probability P(dot sent|dot received).

(d) Compute the probability P(dash sent|dot received).