Department of Mathematics
Comprehensive Examination
2023 Fall Semester \Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes.

Time: 2 hours

Math 640: Complex Variables

No Complex Variables exam was given in Fall 2023

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Math 660: Topology

4. Recall the following de nition

   > De nition: The boundary of A, denoted by $@A$, is de ned to be
   >
   > $$\overline{A} \setminus \overline{X \quad A} :$$

   Let $(X; T)$ be a topological space. Is it true that if $\overline{A} \quad \overline{B}$ then $@A \quad @B$? Give a full proof or provide a counterexample.

5. Let A be a compact subspace of a Hausdor space X. Prove that A is closed.

6. Recall the following de nitions:

   > De nition: Let X be a topological space. Then X is connected if and only if X is not the union of two disjoint non-empty open sets.

   > De nition: Let X be a topological space. Subsets $A; B$ in X [A417([([-417([([y -11.971 0 Tdsd)-3

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Math 675: Differential Equations

7. Consider the system

$$x'(t) = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x(t)$$

(a) Find the general solution to the system. Express your answer in terms of real valued functions.

(b) Determine whether solutions rotate clockwise or counterclockwise. Explain your process.

8. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

The general solution to the homogeneous system $x' = Ax$ is

$$x_{hom} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \qquad (*)$$

(a) What are the eigenvalues and eigenvectors for A? Explain.

(b) Calculate the linear propagator $\exp(A)$ using part (a).

(c) Use Eq. (*) to find the general solution to the inhomogeneous system:

$$x'(t) = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} e^{4t} \\ 2e^{4t} \end{pmatrix}$$

Simplify your answer as much as possible.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Consider the vector valued functions

$$x(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; \quad y(t) = \begin{pmatrix} 2f(t) \\ 3f(t) \end{pmatrix}$$

where $f(t) > 0$ and $f'(t) > 0$ for all $t \in (-\infty, \infty)$.

(a) Fix $t_0 \in (-\infty, \infty)$. Show that the set $S(t) = \{x(t), y(t)\}$ is linearly dependent at each point $t = t_0$.

(b) Prove, directly from the definition, that the set

$$S(t) = \{x(t), y(t)\}$$

is linearly independent on the interval $(-\infty, \infty)$.

(c) Calculate the Wronskian $W[x(t), y(t)]$.

(d) Could $x(t)$ and $y(t)$ be solutions to the same 2D linear system of ODE $x' = Ax$? Explain your thinking.

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Math 680: Optimization

10. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableaux, and state the solution. You can use either fractions or decimals (rounded to the nearest hundredth).

$$\begin{array}{rrrrr}
\text{maximize} & 6x_1 & +8x_2 & +9x_3 & \\
\text{subject to} & 2x_1 & +4x_2 & +2x_3 & 11 \\
 & 3x_1 & 3x_2 & +4x_3 & 28 \\
 & x_1 & +2x_2 & +3x_3 & 30 \\
 & x_1 & +2x_2 & + \;\; x_3 & = 32 \\
 & & & x_1; x_2; x_3 & 0
\end{array}$$

11. Prove or disprove, using the Complementary Slackness Theorem, that (7,0,3) is the optimal solution to

$$\begin{array}{rrrrr}
\text{min} & 5x_1 & +6x_2 & +8x_3 & \\
\text{subject to} & x_1 & +4x_2 & +3x_3 & 16 \\
 & 2x_1 & +3x_2 & 2x_3 & 8 \\
 & 3x_1 & +5x_2 & +2x_3 & 26 \\
 & & & x_j & 0
\end{array}$$

12. Consider the problem

$$\text{min} \quad 6x_1 + 8x_2 + 3x_3$$

Department of Mathematics
Comprehensive Examination
2023 Fall Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1.

Math 630

4. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Prove that if f is Riemann integrable, then for every $\epsilon > 0$, there exists a subdivision P of $[a,b]$ such that

$$U(f;P) - L(f;P) < \epsilon .$$

5. Let $A \subset M$ where M