

Department of Mathematics
Comprehensive Examination
2019 Fall Semester Part 1: Core Classes

Directions:

- You will answer FOUR questions.
- You MUST choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Time: 2.5 hours

Math 620

1. Let G be a group. Prove that G is abelian if and only if the map $\phi : G \rightarrow G$ defined via $\phi(g) = g^{-1}$ is a group automorphism.
2. Let R be an integral domain. Prove that for a in R , $\langle a \rangle = R$ if and only if a is a unit.

Math 630

3. Consider the sequence of functions $f_n(x) = x^{2^n}$.
(a)

Math 670

5. Consider the equation $e^x = 3 - (x - 1)^2$

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2019 Fall Semester Part 2: "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 640: Complex Analysis

1. Let D be an open connected subset of \mathbb{C} and let $f: D \rightarrow \mathbb{C}$ be analytic in D . Prove that if \bar{f} is analytic in D then f is constant in D .

2. Prove: If f is analytic within and on a simple closed contour γ , and z_0 is not on γ , then

$$\int_{z \in \gamma} \frac{f(z)}{(z - z_0)^2} dz = \int_{z \in \gamma} \frac{f'(z)}{z - z_0} dz$$

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2019 Fall Semester Part 2: "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 660: Topology

3. Let X be a Hausdorff space and $f: X \rightarrow X$ be a continuous function.
Prove: $F = \{x \in X : f(x) = x\}$ is closed in X .

4. Let A and B be disjoint compact subspaces of a Hausdorff space X .
Prove: There exist disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

(Recall: For any compact subspace Y of a Hausdorff space X , there exist disjoint open neighborhoods of x_0 and Y .)

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Time: 2 hours

Math 675: Differential Equations

5. Consider the following differential equation

$$(x^2 - 1)y'' + xy' - y = 0$$

- (a) Find the recursion relation for the series solution centered at $x_0 = 0$.
- (b) Find the first six non-zero terms of the series solution centered at $x_0 = 0$. Write your final answer in terms of the coefficients a_0 and a_1 .

6. Consider the system of differential equations given by

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x}$$

- (a) What is the general solution to the homogeneous differential equation above?
- (b) What is the general solution to the inhomogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}.$$

Simplify your answer.

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Time: 2 hours

Math 680: Optimization

7. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{llll} \text{maximize} & 6x_1 & + 9x_2 & + 10x_3 \\ \text{subject to} & 3x_1 & + 2x_2 & - 6x_3 & 24 \\ & x_1 & + 5x_2 & + 2x_3 & 18 \\ & 3x_1 & + 3x_2 & + 4x_3 & = 24 \\ & & & & x_1, x_2, x_3 & 0 \end{array}$$

8. Consider the problem

$$\begin{array}{llll} \text{maximize} & 4x_1 & + 8x_2 & + 3x_3 \\ \text{subject to} & 2x_1 & - 3x_2 & + 2x_3 & 26 \\ & -3x_1 & + x_2 & + 4x_3 & 24 \\ & 3x_1 & + 2x_2 & - 2x_3 & 30 \\ & & & & x_1, x_2, x_3 & 0 \end{array}$$

The first and last tableau are shown below.

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	2	-3	2	1	0	0	26
x_5	-3	1	4	0	1	0	24
x_6	3	2	-2	0	0	1	30
	-4	-8	-3	0	0	0	0
x_1	1	0	0	$\frac{5}{28}$	$\frac{1}{28}$	$\frac{1}{4}$	13
x_3	0	0	1	$\frac{9}{56}$	$\frac{13}{56}$	$\frac{1}{8}$	$\frac{27}{2}$
x_2	0	1	0	$-\frac{3}{28}$	$\frac{5}{28}$	$\frac{1}{4}$	9
	0	0	0	$\frac{19}{56}$	$\frac{127}{56}$	$\frac{27}{8}$	$\frac{329}{2}$

Use sensitivity analysis to answer the questions below. For each situation, return to the original problem as given.

- (a) How much can c_2 , the coefficient of x_2 , change in the objective function and not change the solution of $(x_1, x_2, x_3) = (13, 9, \frac{27}{2})$?
- (b) What would be the new solution if the following constraint is added to the system?

$$x_1 - 2x_2 + 3x_3 = 12$$